Construction of Wavelet Filters

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These ideas are not original, but are not particularly easy to extract from the literature, so I am making this pedagogical note. I will not cite original literature, because it will not be useful, and I will not cite lecture notes, because they are idiosyncratic and have different notation, but I will add a bibliography in this footnote[1].

Daubechies wavelet filters are the essential input to the multiresolution decomposition (MRD). How do we construct these filters?

Let the smoothing wavelet be written $\vec{\phi}$ with entries ϕ_i, and the differencing wavelet be written $\vec{\psi}$ with entries $\psi_i = (-1)^{i+1}\phi_{N-1-i}$ for a filter of length $N \in 2\,\mathbb{Z}$ (and we are using a zero-indexed numbering scheme compatible with Python). The "fundamental" equations necessary for any finite-impulse response (FIR) filter are

$$\sum_{i=0}^{N-1} \phi_i = \sqrt{2}, \qquad \sum_{i=0}^{N-1} \psi_i = 0. \tag{1}$$

Beyond this, Daubechies-N filters have the special feature of having $N/2$ vanishing moments, meaning that they "annihilate" polynomials of order $N/2-1$. So the Daubechies-2 filter, which is also called the Haar filter, only sets a constant to zero when it takes differences, whereas a Daubechies-4 filter sets linear polynomials to zero. (This is very cool!) The condition that allows this to be true provides $N/2$ equations. Similarly, the Daubechies-N filters are orthonormal, furnishing $N/2$ additional equations. Together these additional conditions are:

$$\sum_{k=0}^{N-1} (-1)^k k^m \phi_k = 0, \qquad \sum_{k=0}^{N-1-2m} \phi_k \phi_{k+2m} = \delta_{0m} \qquad \text{for } m = 0, 1, \ldots \frac{N}{2} - 1. \tag{2}$$

Note that the first equation with $m = 0$ is the same as the fundamental equation for the differencing coefficients, $\sum_{i=0}^{N-1} \psi_i = 0$. Thus, we have $N+1$ constraints relating N values. This is an overdetermined set of equations, but it is possible to find solutions.

To take an example, let's consider $N = 4$. We have

$$\phi_0 + \phi_1 + \phi_2 + \phi_3 = \sqrt{2} \tag{3a}$$
$$\phi_0 - \phi_1 + \phi_2 - \phi_3 = 0 \tag{3b}$$
$$-\phi_1 + 2\phi_2 - 3\phi_3 = 0 \tag{3c}$$
$$\phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 = 1 \tag{3d}$$
$$\phi_0\phi_2 + \phi_1\phi_3 = 0 \tag{3e}$$

[1] The single most useful reference I have found is this one. The original papers are of course Mallat's thesis from 1989 and Daubechies' work from the late 80s, culminating in her book from 1991.

Eq. (3a) is the first "fundamental" equation in Eq. (1), Eqs. (3b) and (3c) come from the first set of equalities in Eq. (2), and Eqs. (3d) and (3e) come from the second set of equalities in Eq. (2). From Eq. (3c) we immediately have $\phi_1 = 2\phi_2 - 3\phi_3$, and we can plug this into Eq. (3b) to get $\phi_0 = \phi_2 - 2\phi_3$. Plugging both of these into Eq. (3a) gives $\phi_2 = \phi_3 + \sqrt{2}/4$. Now plugging all of this into Eq. (3e), we have

$$\begin{aligned} 2\phi_3^2 - \frac{1}{\sqrt{2}}\phi_3 - \frac{1}{8} = 0 \implies \phi_3 = \frac{1 \pm \sqrt{3}}{4\sqrt{2}} \implies& \\ \implies \{\phi_0, \phi_1, \phi_2, \phi_3\} = \frac{\{1-\sqrt{3}, 3-\sqrt{3}, 3+\sqrt{3}, 1+\sqrt{3}\}}{4\sqrt{2}}& \end{aligned} \tag{4}$$

One can check that this solves Eq. (3d).